ACTIVITIES THAT MATHEMATICS MAJORS USE TO BRIDGE THE GAP BETWEEN INFORMAL ARGUMENTS AND PROOFS

Dov Zazkis, Keith Weber, Juan Pablo Mejia-Ramos

Rutgers University

In this paper we examine a commonly suggested proof construction strategy from the mathematics education literature—that students first produce an informal argument and then use this as a basis for constructing a formal proof. The work of students who produce such informal arguments during proving activities was analyzed to distill three activities that contribute to students' successful translation of informal arguments into formal proofs. These are elaboration, syntactification, and rewarranting. We analyze how attempting to engage in these activities relates to success with proof construction. Additionally, we discuss how each individual activity contributes to the translation of an informal argument into a formal proof.

INTRODUCTION

Proving is central to mathematical practice. Unfortunately, numerous studies have documented that mathematics majors struggle with proof construction tasks and have documented numerous causes for these difficulties (see Weber, 2003, for a review of some difficulties). However, research on how mathematics majors can or should successfully write proofs has been comparatively sparse. In this paper, we examine one suggestion from the literature—that students base their formal proofs on informal arguments (e.g., Garuti, Boero, & Lamut, 1998; Raman, 2003; Weber & Alcock, 2004).

THEORETICAL PERSPECTIVE

Basing proofs on informal arguments

Boero (1999) observed that a proof must satisfy certain formal constraints, but the reasoning used to generate this proof need not. In particular, the informal arguments that one uses to understand why a proposition is true can be used as a basis for constructing a proof of this proposition (e.g., Bartollini-Bussi et al., 2007; Garuti, Boero, & Lamut, 1998). A number of researchers have advocated that students base their proofs on informal arguments. This is a driving force behind the research program of the Italian school whose proponents endorse proofs having a cognitive unity where, under particular circumstances, there is a continuum between a student's production of a conjecture and how the student proves it (e.g., Garuti, Boero, & Lamut, 1998; Pedemonte, 2007). Support for these recommendations typically comes from the analysis of students successfully basing proofs off of informal arguments (e.g., Garuti, Boero, & Lamut, 1998) and that this is common in authentic mathematical practice (Raman, 2003).

2014. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 5, pp. 417-424. Vancouver, Canada: PME.

To distinguish between an informal argument and a proof in advanced mathematics, we follow Stylianides (2007) who proposed assessing whether an argument is a proof along three criteria: (i) *the representation system* (as opposed to proofs, informal arguments may be expressed in terms of graphs or imprecise language), (ii) *the facts that are taken as the starting points* (in proofs, unjustified statements must be accepted by the mathematical community as true whereas in informal arguments, the individual only needs to believe they are true), and (iii*) inference methods* (the methods employed in a proof must be considered valid by one's mathematical community, whereas in an argument the methods of inference must merely be *plausible* to the individual).

Research on bridging the gap between argumentation and proof

In recent years, researchers concerned about the gap between informal arguments and proofs have begun to look at how this distance is traversed. Much of the research can be divided into two categories: analyzing the types of arguments that are easier to translate into proofs and designing classroom environments that help bridge this gap.

In the first category, researchers such as Pedemonte have conceptualized the *distance* between the informal arguments and the corresponding formal proofs (e.g., Pedemonte, 2002, 2007). Pedemonte observed that if the general method of inference (structural distance) or the mathematical ideas (content distance) used in an informal argument and the corresponding proof differ greatly, students will face difficulties in writing the proof (e.g., Pedemonte, 2002, 2007). The second category of studies examines instructor roles in helping students build proofs of informal arguments. This includes research on creating instructional environments (e.g., Bartollini-Bussi et al., 2007) and teacher moves that may facilitate this behavior (e.g., Stylianides, 2007).

In this paper we explore how mathematics majors bridge the gap between informal arguments and proofs by addressing the following two questions: (1) What activities do mathematics majors engage in when they successfully write a proof based on an informal argument? (2) To what extent can these activities account for their success? The answer to these questions can inform instruction by highlighting what skills and practices students need to learn to write proofs based on informal arguments.

METHODS

We recruited 73 mathematics majors from a large public university in the United States who had recently completed a second linear algebra course. Participants met individually with an interviewer for two sessions, each session lasting approximately 90 minutes. In one session, the participants worked on 7 proof construction tasks in linear algebra; in the other, they completed 7 proof construction tasks in calculus. In each session, participants were presented with proving tasks that could be approached either syntactically or semantically (in the sense of Weber & Alcock, 2004). Participants were asked to "think aloud" as they completed each task, given 15 minutes per task and told to write up a proof as if they were submitting it on a course exam. This corpus yielded a total of 1022 proof attempts (73×14).

We coded a participant's argument as informal whenever it was a multi-inference argument where at least one of the inferences was drawn from the appearance of a graph or a diagram, or the inspection of a specific example. There were 37 informal arguments of this type in our data set. In this paper, we focus on these arguments, and how students attempted to translate these arguments into proofs.

ANALYSIS

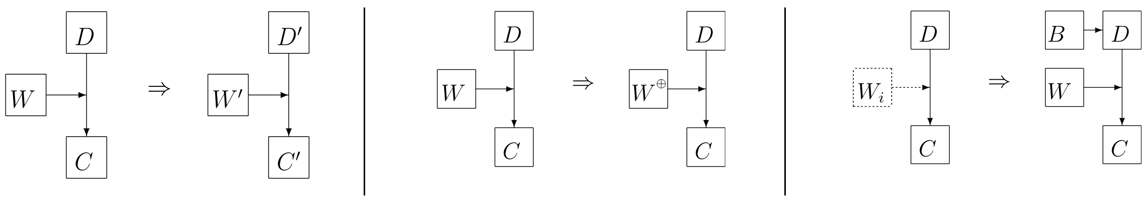
Two research assistants, who are not authors of this paper, coded each proof as valid or invalid. There was 96% agreement on their codings across the data set. Among the 37 proof attempts considered, 14 were coded as valid and 23 were coded as invalid.

Following Pedemonte (2007), we used the basic Toulmin (2003) scheme to analyze each inference that the participant drew in his or her informal argument and final proof. According to the basic Toulmin (2003) scheme, each inference (or sub-argument) contains three parts, the claim (C) being advanced, the data (D) used to support the claim, and the warrant (W) that dictates how the claim follows from the data. In many cases, a warrant was not explicitly stated. In these cases, if possible, we would infer the warrant that the participant was using. This allowed us to notice differences between the participant's initial informal argument and their final proof.

For the 14 successful proof attempts, we used an open coding scheme in the style of Strauss and Corbin (1990) to categorize the ways that the mathematics majors attempted to transform their argument into a proof. This process yielded three categories of activity: *syntactifying, rewarranting*, and *elaborating*. Once these categories were created and defined, we then went through each of the 37 proof attempts, seeking out evidence of participants' attempts to engage in these activities.

RESULTS

In this section, we describe syntactifying, rewarranting, and elaborating, which we illustrate graphically using Toulmin's scheme in Figure 1.



(a) Syntactifying (b) Rewarranting (c) Elaborating

Figure 1: Three translation activities

Syntactifying

Syntactifying occurred when a participant attempted to take a statement in the informal argument that is given in what are perceived to be non-rigorous terms and translate it into what is considered to be a more appropriate representation system for proofs. Such actions included removing references to a diagram used in the informal argument and replacing them with more conventional mathematical terminology, or introducing

algebraic or logical notation. In terms of Toulmin's scheme, we can regard syntactifying as translating the data (D), claim (C), and/or warrant (W) of an argument into new data (D'), claim (C'), and/or warrant (W') in another representation system, without intending to change the meaning of D, C, or W. We illustrate this with Figure 1a. The following informal argument occurred in student A's work on when proving the derivative of a differentiable even function is odd.

Student A: Okay, Like okay, since it's symmetric about the *y*-axis, so it's like a mirror and then all the tangent lines, all the derivatives would be like some values [pointing at negative side of x^2 graph] and then this would just, since it's a mirror would be the negative of them [pointing at positive side of x^2 graph]. So it would be odd.

In the above excerpt Student A argues that since even functions are symmetric about the *y*-axis (D) the *y*-axis acts like a mirror (C). This mirror property is then used as data to justify that $-f'(a) = f'(-a)$ for all *a*. The warrants used are implicit and perceived visually from the graph of x^2, which is used as an example of a generic even function. Later, Student A syntactifies parts of this argument when she shifts away from discussing tangents in terms of the graph.

Student A: How do I put that into words? [...] This is what we want *f* prime of negative *x* equals negative *f* prime of *x*. [writes *f'(-x)=-f'(x)*]. Okay, so if we take the derivative at negative, this would be the negative of *f* of *x*'s derivative, which makes sense. So how do we get from *f* of negative *x* equals *f* of *x* [writes *f(-x)=f(x)*]. Use the definition? Okay lets try that. *f* prime *x*. So by the definition of derivative its like as this approaches this point then the tan line of that. This is the limit at *a*. Either way, *f* of *x* minus *f* of *a*. over *x* minus *a*. [writes $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$].

Student A first syntactifies the end points of the argument. She begins with the conclusion, stating that she is trying to show that *f'(-x)=-f'(x)*. This is a syntactification of her claim that the derivatives on one side are the negation of the other. She then syntactifies the initial data when she uses the analytic definition of odd (*f(-x)=f(x)*) to replace the graphical definition used in her informal argument. Although the chain rule can be used to warrant going directly from the data to the claim, she instead begins to build a proof based on her informal argument. Student A's use of the definition of limit at a point can be seen as a syntactification of the tangent part of her argument, since the limit definition is used to find the slope of a tangent at an arbitrary point. By syntactifying she has moved from working with a graphical representation to an analytic representation and in doing so has shifted to a more appropriate representation system for a proof. The completion of her argument is discussed in the subsequent section on rewarranting.

Rewarranting

Many informal arguments employ warrants that are not permissible in a proof. Rewarranting occurred when the participant tried to find a deductive reason for a claim that their informal argument justified in a non-deductive manner. In terms of Toulmin's scheme, we can regard rewarranting as replacing a *plausible warrant* (W) (i.e., a warrant that the participant believes is likely to yield truth) with a *valid warrant* ($W^{\oplus}$) (i.e., a warrant that the participant believes is considered valid by the mathematical community). This is illustrated in Figure 1b. This differs from syntactifying a warrant (W $\Rightarrow$W'), since this involves expressing W more formally but without changing its meaning. Below is the continuation of Student A's work on the odd/even problem from the previous section, syntactification.

Student A: Since *f* of *x* is even then *f* of negative *x* is equal to negative *x*. Now limit as *x* approaches *a* of tangent. I guess that is the right... consider *a*. Then *f* prime of [mumbling]. So *a* should be... then...is equal to some L. [writes $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=L$] so *f* prime of negative *a* equal but this is the same thing as *f* of *x* minus *f* of *a*. over *x* minus minus *a*. [$f'(-a)=\lim_{x\to a}\frac{f(x)-f(-a)}{x-(-a)}$] Which somehow equals negative L. *f* prime of negative *x* equals negative *f* of [writes silently $f'(-a)=\lim_{x\to a}\frac{f(x)-f(-a)}{x-(-a)}=\lim_{x\to a}\frac{f(x)-f(a)}{a-x}=-L$, so *f'(-x)=-f'(x)*].

In the above excerpt student A algebraically manipulates the limit definition of derivative at a point to show that *f'(-x)=-f'(x)*. This changes the nature of the warrant that links the data that *f(-x)=f(x)* to the claim that *f'(-x)=-f'(x)*. The warrant in the original informal argument based on the visual appearance of a graph. It is replaced here by a string of algebraic manipulations. The new warrant is not simply a translation of the previous warrant that leaves the meaning of the warrant unchanged; it is a different route to linking the data and claim.

Elaborating

Elaborating occurred when a participant attempted to add more detail to the proofs that were not present in their informal argument. This occurred in several ways: Participants would justify statements that they took for granted in their informal arguments by making explicit warrants that were initially implicit (W_i) in their informal arguments, or further justify their data (D) (i.e., the participant attempted to justify a fact that was taken for granted). We illustrate this in Figure 1c. The example below is of the first type, justifying claims initially taken for granted. It occurred during a participants work on the problem: Prove that $\int_{-a}^{a}\sin^3(x)dx=0$ for any real number *a*.

Student B: Um it [$sin^3(x)$] must be an odd function. […] Right it'll be symmetrical across the identity line, which would mean that the integral from negative *a* to zero should be the negation of zero to *a*. And so it would be zero.

In this excerpt the participant has an informal argument that $\int_{-a}^{a} \sin^3(x)dx = 0$. Notice that within this argument the assertion that $sin^3(x)$ is odd is treated as a known fact (data). Immediately following this informal explanation the participant begins to *elaborate* this argument by providing a justification for this assertion.

Student B: I'm trying to think how to show that sin of *x* cubed is odd. So basically I'd have to show that *f* of negative *x* has to equal negative *f* of *x*. Is that right... yes. So sin cubed of negative *x*... sine by definition is an odd function [writes $sin(-x)=-sin(x)$]. Uh Yeah. So sin cubed negative is equal to sin negative *x* times sin negative *x* which is equal to sin of *x* times sine of *x* times sin of *x*. Which is sin of *x* cubed. Quantity cubed. [writes: $sin^3(-x)=sin(-x)sin(-x)sin(-x)=(-sin(x))(-sin(x))(-sin(x))=-sin^3(x)$] So it's odd.

In the above excerpt, what was originally taken as data (D) in the argument ($sin(x)$ is odd) is now taken to be the claim (C) of a new sub-argument. Student B shifts the starting point for the proof from $sin^3(x)$ is odd to $sin(x)$ is odd, which is arguably more mathematically appropriate.

A student may also elaborate by replacing an implicit warrant in their informal argument with an explicit one in their formal proof. The following excerpt is taken from student C's work on the problem: Suppose $f(0)=f'(0)=1$ and $f''(x)>0$ for all positive *x*. Prove that $f(2)>2$).

Student C: If the second derivative is greater than zero then *f* prime of *x* is increasing.

So we know that *f* prime of zero equals one [draws: ]. So the derivative at zero equals one and the derivative is always increasing then the slope is greater than one after zero. Which means *f* of 1 is greater than one and *f* of 2 is greater than two. Well it makes sense.

In the above student C produced an informal argument that relied on a graph. Notice that he, among other things, argues that $f'(x)$ is increasing and $f'(0)=1$ (D) implies that $f'(x)>1$ for $x>0$ (C). The implicit warrant here is the definition of increasing. Later when he writes a formal proof this warrant is no longer implicit:

Student C: [saying what he writes] If *f* double prime of *x* greater than zero, then *f* prime *x* is increasing for all positive *x*. Thus for any *x* sub 1 comma *x* sub two in the interval zero to infinity such that *x* sub 2 is greater than *x* sub 1. *f* prime of *x* sub 2 is greater than *f* prime of *x* sub 1. *f* prime of zero equals one. Thus *f* prime of *x* sub 2 is greater than *f* prime of *x* sub 1 is greater than one. The derivative at any point greater than zero is greater than 1…

Notice that in his proof he explicitly uses the formal definition of increasing ($x_2>x_1 \Leftrightarrow f(x_2)>f(x_1)$), which was an implicit warrant in the informal argument. So elaboration has occurred. However, even though the proof involves taking smaller steps than the informal argument, the path the reasoning follows is unchanged.

Prevalence of these three activities

In Table 1, we present the frequency with which a participant attempted to engage in these activities as a function of whether or not they were able to successfully produce a proof. As Table 1 illustrates, participants who successfully produced proofs were considerably more likely to engage in syntactifying, rewarranting, and elaborating. Those who were successful in writing a proof usually engaged in all three activities, while those who were not successful rarely engaged in all three.

	Total	Syntactifying	Rewarranting	Elaborating	All three
Successful	14	12 (85%)	12 (85%)	11 (79%)	11 (79%)
Unsuccessful	23	15 (65%)	9 (39%)	12 (52%)	4 (17%)

Table 1: Attempted engagement with translation activities and success

Slicing the data another way, there were 15 instances in which a participant engaged in all three activities, and they succeeded in writing a proof 11 times (73% of the time). Among the 22 instances in which a participant did not engage in all three activities, the participants only succeeded in writing a proof three times (14% of the time); in two of those successful instances, the proofs produced differed substantially from the informal argument

It is important to note that Table 1 documented whether a participant *attempted to engage in the activity*, not if this engagement was successful. Consequently, we believe a key factor in determining success in proof writing for these participants was their willingness to try to syntactify, rewarrant, and elaborate.

DISCUSSION

The data in this paper contributes to the literature on bridging the gap between informal arguments and proofs. We highlighted three activities—syntactifying, rewarranting, and elaborating—that contribute to writing a proof based on an informal argument. Syntactifying is used to translate data, claims and/or warrants stated in terms of informal representations and natural language to the representation system of proof. If successful, this results in an argument that uses the appropriate representation system. Elaborating adds additional details to an argument by shifting the starting point of an argument to a more basic and widely accepted statement and making clear how new inferences were derived. Rewarranting seeks to replace plausible warrants with valid ones, changing the meaning of the argumentation into one more acceptable for proof.

We observed that there was a relative scarcity of informal arguments produced across this large data set (37 instances across 1022 proof attempts). In this respect, we support

research into the design of instructional environments that encourages students to create proofs based on these informal arguments (e.g., Bossulini-Bussi et al., 2007). We also observed that participants who engaged in syntactifying, rewarranting, and elaborating once their informal arguments were produced enjoyed far greater success in proof-writing than those who did not. Consequently, we hypothesize that some of students' difficulties with bridging the gap between informal arguments and proofs is due to students' inability to successfully engage in these activities. Designing instruction that specifically targets these activities has the potential to improve mathematics majors' abilities to write proofs and would be a useful direction for future research.

References

Bartolini Bussi, M., Boero, P., Ferri, F., Garuti, R., & Mariotti, M. (2007). Approaching and developing the culture of geometry theorems in school. In P. Boero (Ed.), *Theorems in school: From history, epistemology, and cognition to classroom practice* (pp. 211-217). Rotterdam: Sense Publishing.

Boero, P. (1999, July/August). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*.

Garuti, R., Boero, P., & Lemut, E. (1998). Cognitive unity of theorems and difficulty of proof. In A. Olivier & K. Newstead (Eds.), *Proc. 22nd Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 345-352). Stellenbosh, ZA: PME.

Pedemonte, B. (2002). Relation between argumentation and proof in mathematics: cognitive unity or break. In J. Novotná (Ed.), *Proceedings of the 2nd Conference of the European Society for Research in Mathematics Education* (Vol. 2, pp. 70-80). Czech Republic: ERME.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, *66*(1), 23-41.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, *52*, 319-325.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. London: SAGE.

Stylianides, A. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*, 289-321.

Toulmin, S. (2003). *The uses of argument*. Cambridge: Cambridge University Press.

Weber, K. (2003). Students' difficulties with proof. In A. Selden and J. Selden (Eds.), *Research Sampler, 8*. Available from http://www.maa.org/t_and_l/sampler/research_sampler.html.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*, 209-234.